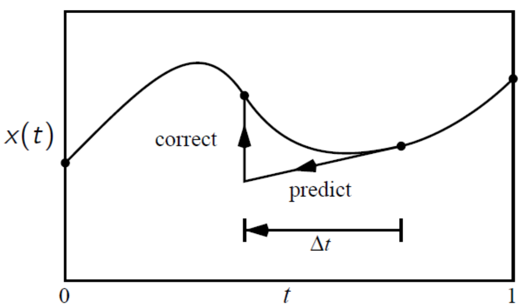
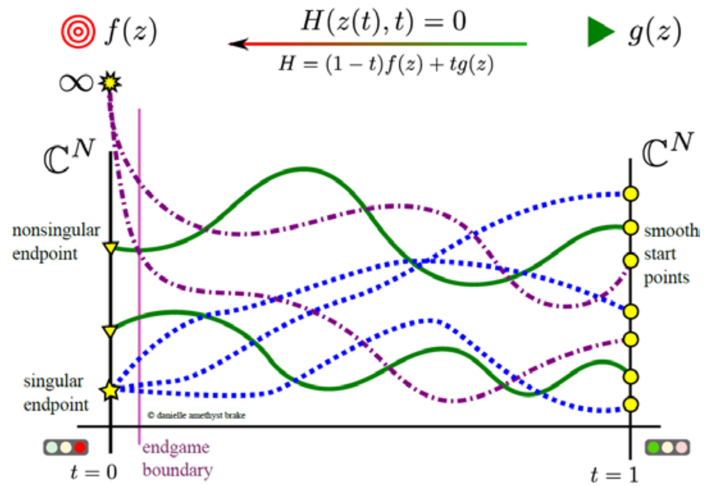
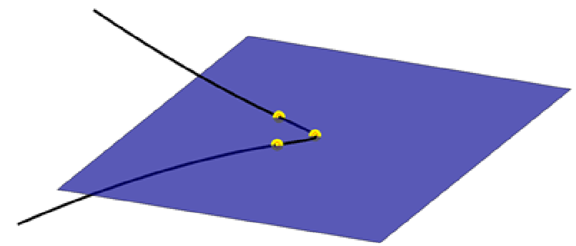
Foundations of Numerical Algebraic Geometry





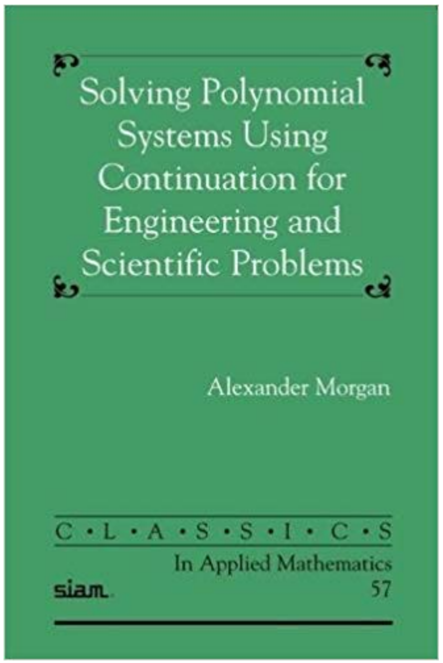


Jonathan Hauenstein

Nonlinear Algebra Bootcamp

September 10, 2018

Overview



Solving Polynomial Systems Using Continuation for Engineering and Scientific Problems
Alexander Morgan
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In Applied Mathematics
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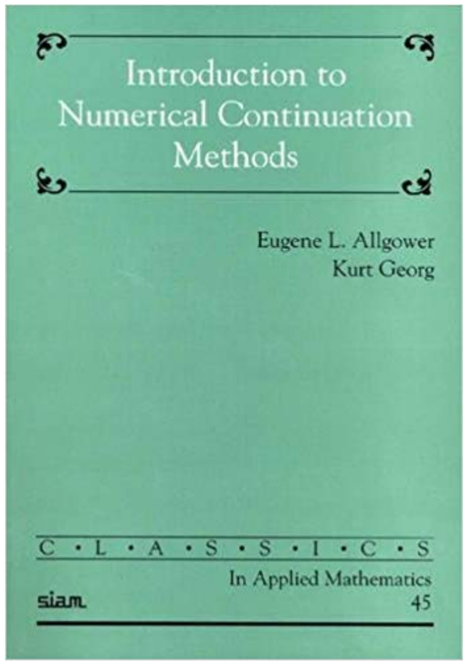

Introduction to Numerical Continuation Methods
Eugene L. Allgower
Kurt Georg
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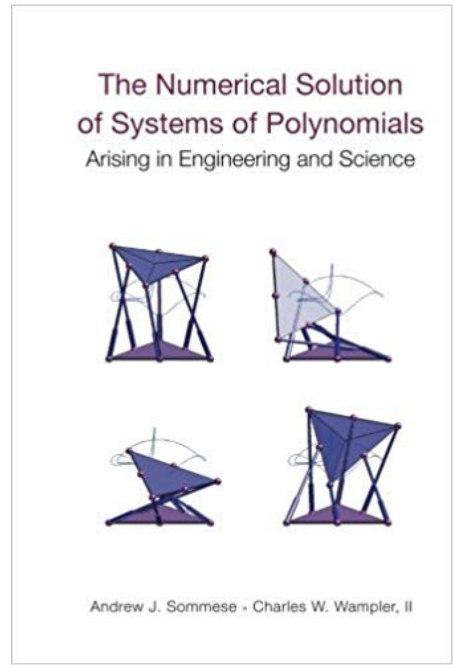
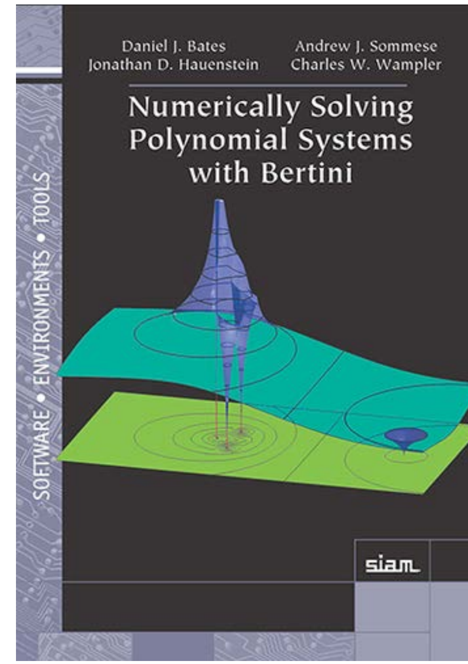


The Numerical Solution of Systems of Polynomials
Arising in Engineering and Science
Andrew J. Sommese • Charles W. Wampler, II



Daniel J. Bates
Jonathan D. Hauenstein
Andrew J. Sommese
Charles W. Wampler
Numerically Solving Polynomial Systems with Bertini
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Overview

For a polynomial system $f : \mathbb{C}^n \to \mathbb{C}^N$, **solve** $f(x) = 0$.

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Maple

```
> solve(x^5 - x + 1);
```

$RootOf(\_Z^5 - \_Z + 1, index = 1), RootOf(\_Z^5 - \_Z + 1, index = 2), RootOf(\_Z^5 - \_Z + 1, index = 3),$
$RootOf(\_Z^5 - \_Z + 1, index = 4), RootOf(\_Z^5 - \_Z + 1, index = 5)$

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> $fsolve(x^5 - x + 1);$

-1.167303978

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> $fsolve(x^5 - x + 1);$

$$-1.167303978$$

> $evalf(solve(x^5 - x + 1));$

$0.764884433600585 + 0.352471546031726\,I, -0.181232444469875 + 1.08395410131771\,I, -1.16730397826142, -0.181232444469875 - 1.08395410131771\,I, 0.764884433600585 - 0.352471546031726\,I$

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Maple

> $evalf\left(solve\left(x^5 - x + 1\right)\right);$

$0.764884433600585 + 0.352471546031726\,I,\ -0.181232444469875 + 1.08395410131771\,I,$
$-1.16730397826142,\ -0.181232444469875 - 1.08395410131771\,I,\ 0.764884433600585$
$-\ 0.352471546031726\,I$

Bertini

input

```
variable_group x;
function f;
f = x^5 - x + 1;
```

finite_solutions

```
5

7.648844336005847e-01 -3.524715460317264e-01

7.648844336005849e-01 3.524715460317262e-01

-1.812324444698754e-01 1.083954101317711e+00

-1.167303978261419e+00 -2.220446049250313e-16

-1.812324444698754e-01 -1.083954101317711e+00
```

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What does it mean to **solve**? Some examples include:

- Show that a solution exists
  - Computing upper bounds on rank of a tensor

    Strassen (1969): $\mathrm{rank}\, M_2 \leq 7$ showing $\omega \leq \log_2 7 < 3$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} a\cdot A + b\cdot C & a\cdot B + b\cdot D \\ c\cdot A + d\cdot C & c\cdot B + d\cdot D \end{bmatrix} = \begin{bmatrix} I + IV - V + VII & III + V \\ II + IV & I - II + III + VI \end{bmatrix}$$

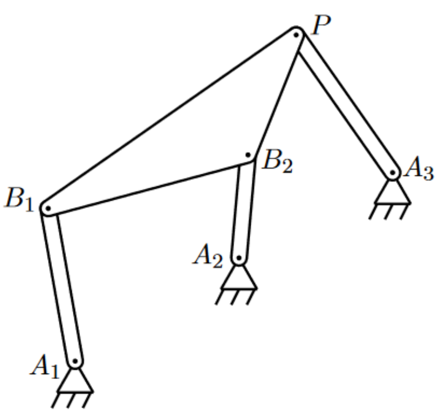
$$\begin{array}{rl} I: & (a+d)\cdot(A+D) \\ II: & (c+d)\cdot A \\ III: & a\cdot(B-D) \\ IV: & d\cdot(C-A) \end{array} \qquad \begin{array}{rl} V: & (a+b)\cdot D \\ VI: & (c-a)\cdot(A+B) \\ VII: & (b-d)\cdot(C+D) \end{array}$$

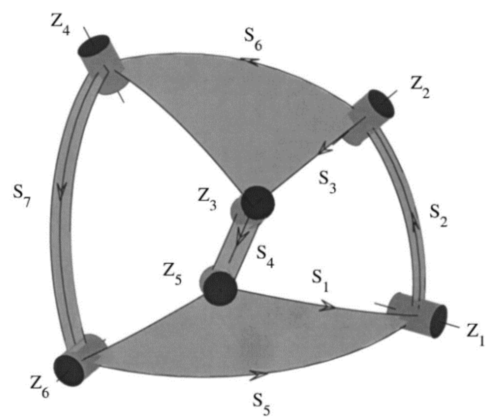
Overview

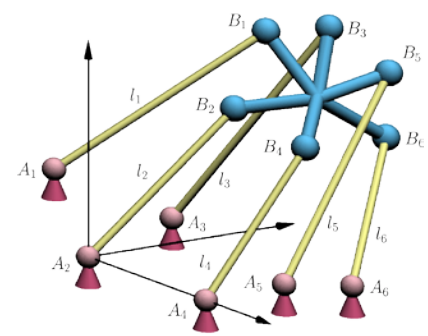
For a polynomial system $f : \mathbb{C}^n \to \mathbb{C}^N$, **solve** $f(x) = 0$.

What does it mean to **solve**? Some examples include:

- Compute all isolated solutions (over $\mathbb{C}$ or $\mathbb{R}$).
  - Number of assembly configurations







planar pentad
SE(2)

spherical pentad
SO(3)

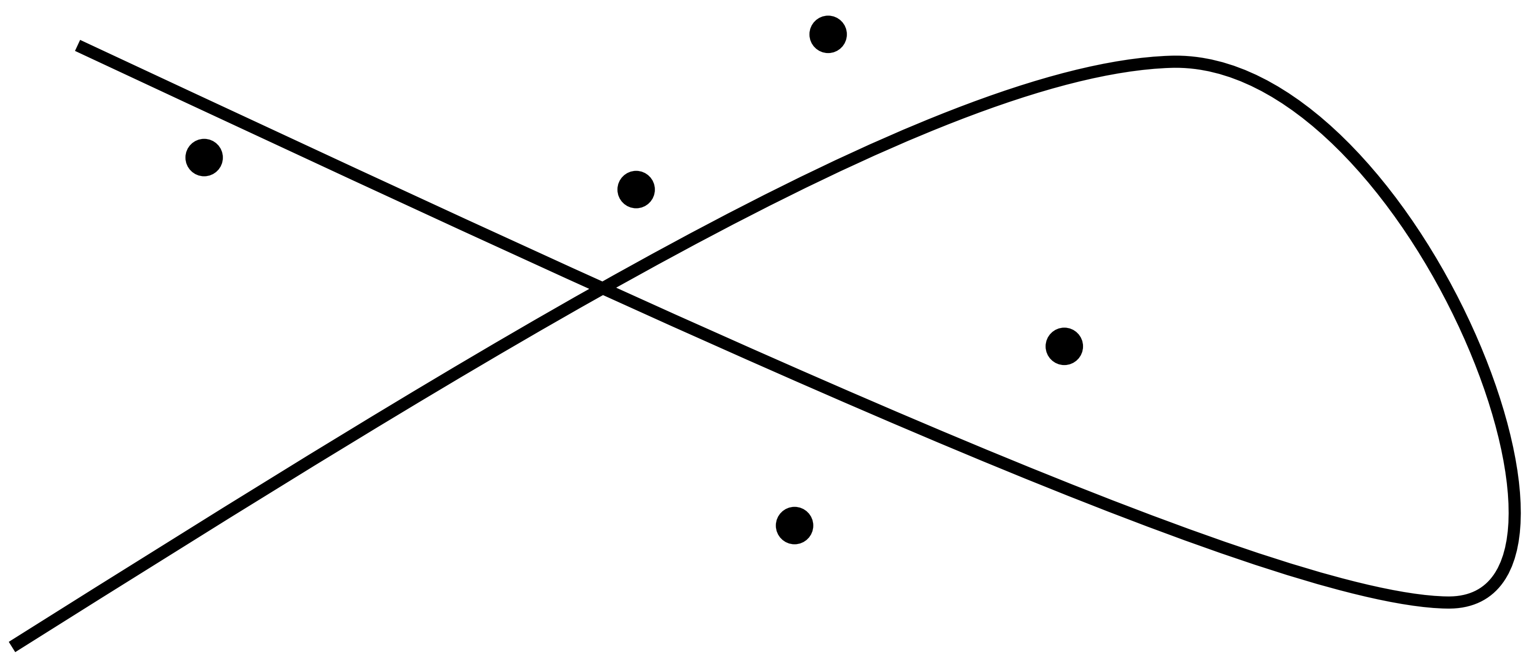
Stewart-Gough plaftorm
SE(3)

Overview

For a polynomial system $f : \mathbb{C}^n \to \mathbb{C}^N$, **solve** $f(x) = 0$.

What does it mean to **solve**? Some examples include:

- Describe all irreducible components.



Overview

For a polynomial system $f : \mathbb{C}^n \to \mathbb{C}^N$, **solve** $f(x) = 0$.

Generally speaking:

- Algebraic methods prefer vastly over-determined systems
  - fewer "new" polynomials to compute
  - Bardet-Faugere-Salvy (2004)

- Numerical algebraic geometry prefers well-constrained systems of low degrees with coefficients of roughly unit magnitude
  - codimension = # equations
  - stable under perturbations

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What does it mean to **numerically solve**?

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What does it mean to **numerically solve**?

Need two key aspects:

- compute sufficiently accurate numerical approximation
- have an algorithm that can produce approximations of solution to any given accuracy starting from numerical approximation
  - *sufficiently accurate* depends on the algorithm

Overview

What does it mean to **numerically solve**?

- compute sufficiently accurate numerical approximation
- have an algorithm that can produce approximations of solution to any given accuracy starting from numerical approximation

Example

$f(x) = x^2 - 2 = 0$

- $x_0 = 1$ is *numerical solution* associated with Newton's method

Overview

Example

$$f(x) = x^2 - 2 = 0$$

- $x_0 = 1$ is *numerical solution* associated with Newton's method

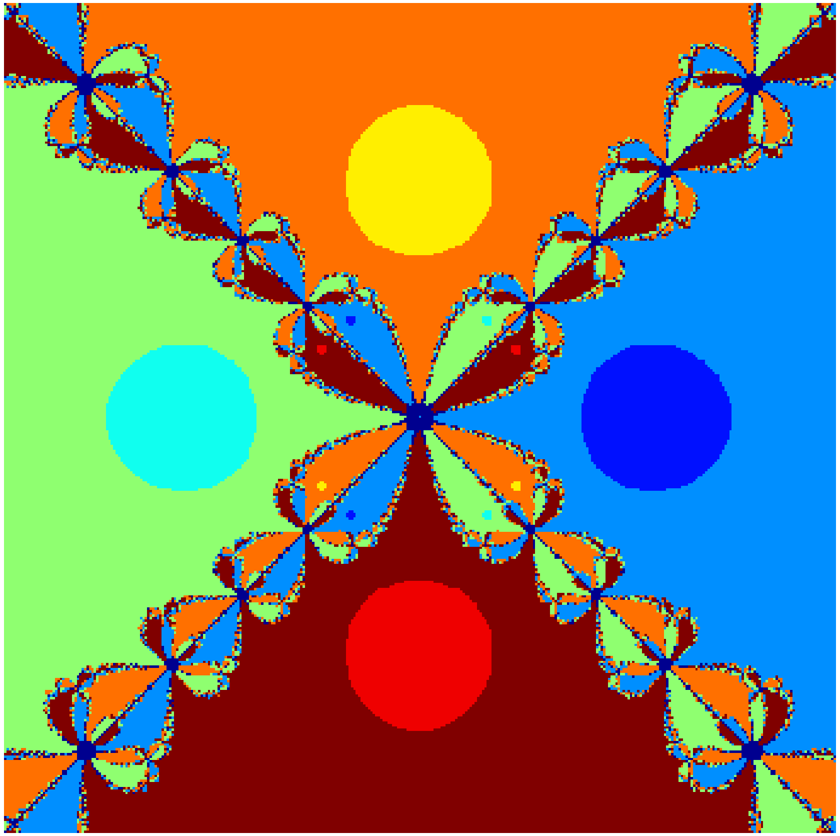
$$x_{k+1} = x_k - Jf(x_k)^{-1} f(x_k)$$

$$\begin{aligned}
x_0 &= 1 \\
x_1 &= 1.5 \\
x_2 &= 1.416666666666666666666666666666666666666667 \\
x_3 &= 1.414215686274509803921568627450980392156862745098\mathrm{0} \\
x_4 &= 1.414213562374689910626295578890134910116559622115\mathrm{7} \\
x_5 &= 1.414213562373095048801689623502530243614981925776\mathrm{2} \\
x_6 &= 1.414213562373095048801688724209698078569671875377\mathrm{2} \\
\sqrt{2} &= 1.414213562373095048801688724209698078569671875376\mathrm{9}
\end{aligned}$$

Overview

Double-edged sword of Newton's method:

- Qudaratic convergence near nonsingular solutions
- Slow convergence or divergence near singular solutions
- Difficulty away solutions (chaos, limit cycles, etc)

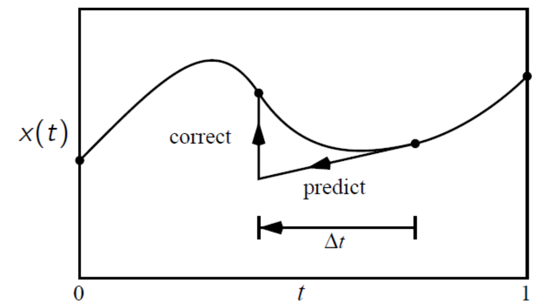


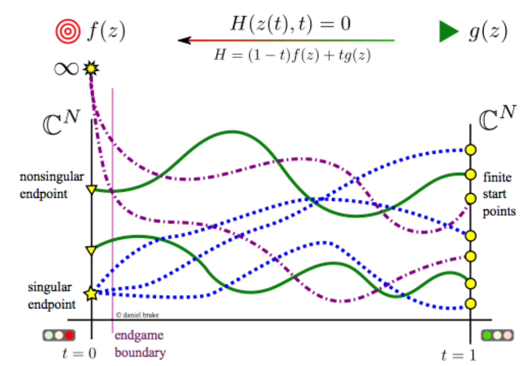
$$f(x) = x^4 - 1$$

Overview

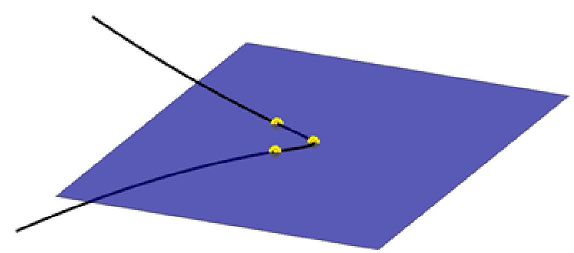
Foundations of Numerical Algebraic Geometry:

- Continuation and path tracking
  - Constructing homotopies





- Witness sets
  - Numerical Irreducible Decomposition
  - Other computations using witness sets



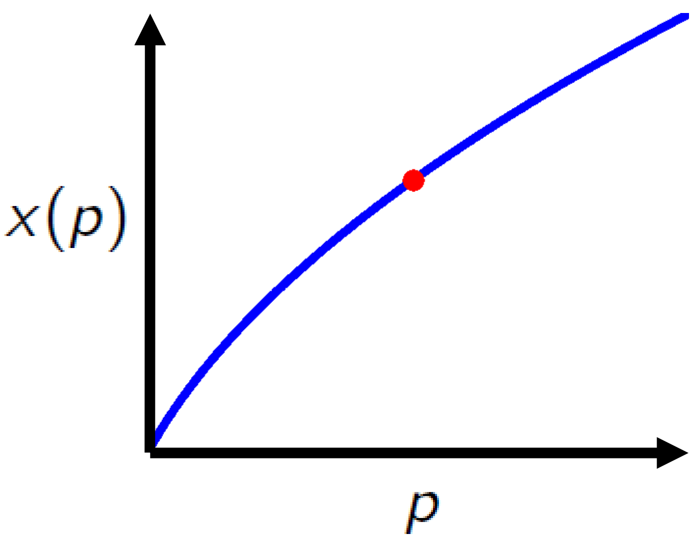
Continuation

Continuation from complex analysis:

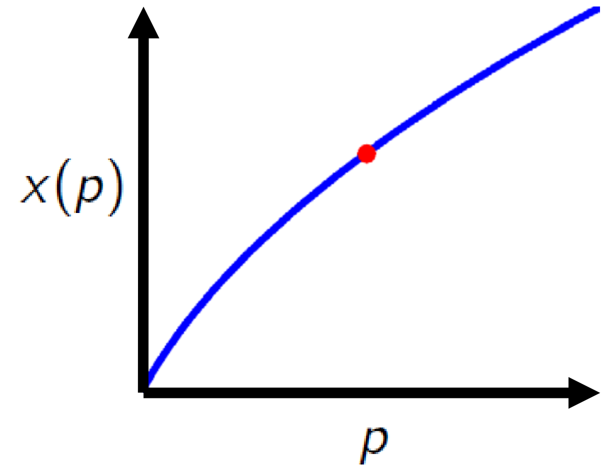
- Cauchy (1789-1857), Riemann (1826-1866), Mittag-Leffler (1846-1927)
- Implicit function theorem
- Analytic extension of functions (analytic continuation)

Big picture idea:

- solutions “continue” locally under small parameter changes



Continuation



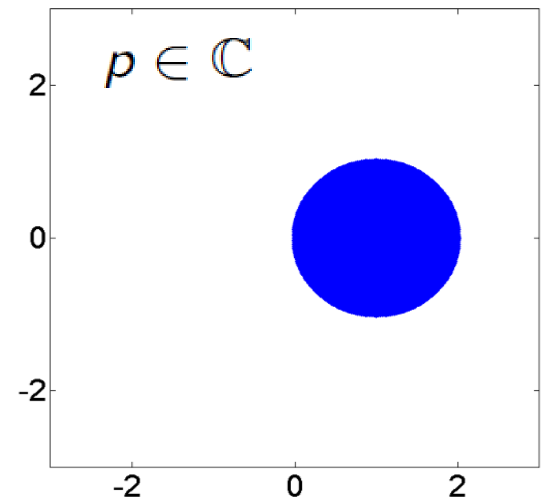
Example

$$f(x;p) = x^2 - p = 0$$

Locally near $p = 1$:

$$x(p) = \sqrt{p} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{4^n (1-2n)(n!)^2} (p-1)^n$$

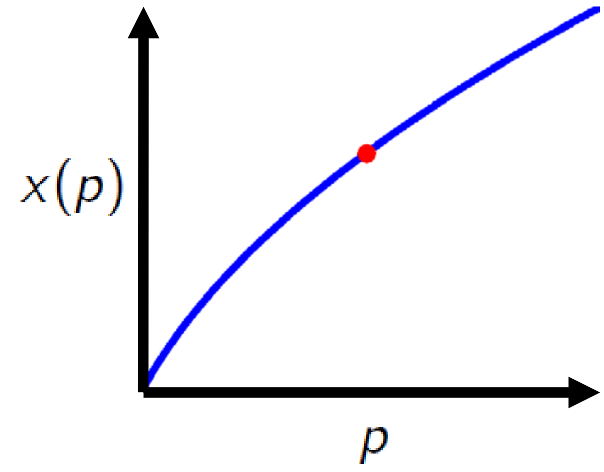
- converges for $|p-1| \leq 1$



Continuation

Example

$$f(x;p) = x^2 - p = 0$$



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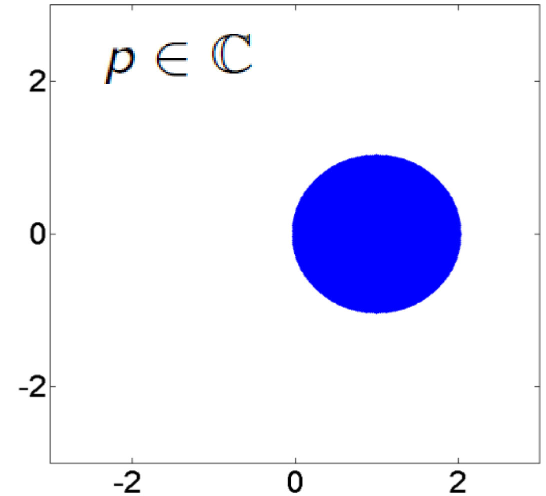


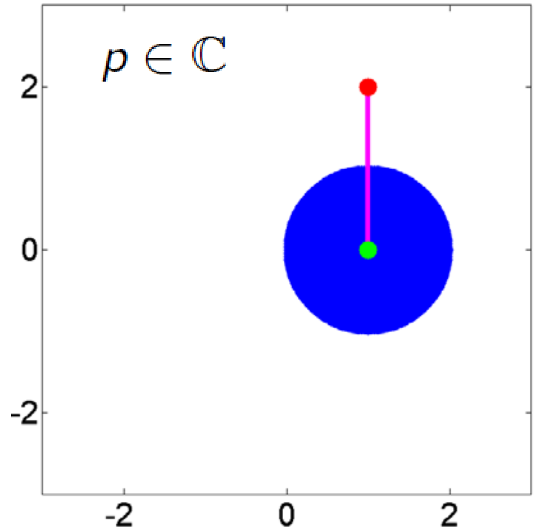
Use continuation to extend beyond this domain.

Continuation

Example

$$f(x;p) = x^2 - p = 0$$

Continue the solution $x = 1$ at $p = 1$ to $p = 1 + 2i$.



Continuation

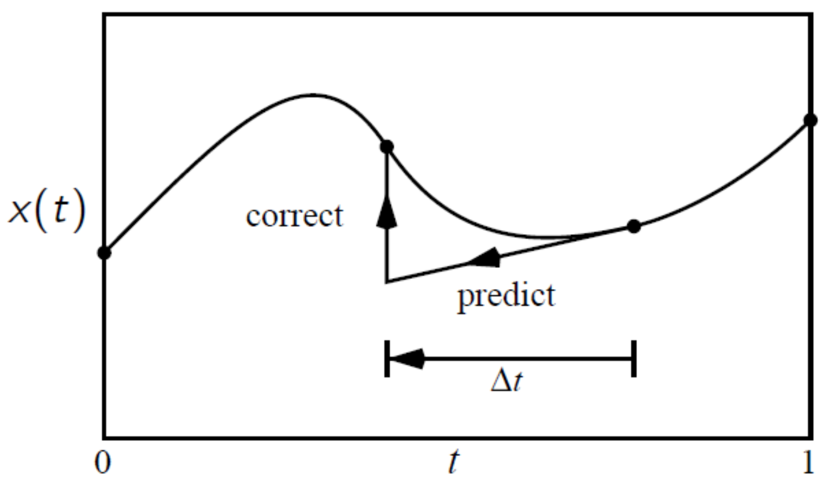
Numerically track along the path $x(t)$ satisfying $H(x(t), t) = 0$:

- (Predictor) Estimate $x(t + \Delta t)$ from $x(t)$ by discretizing using the Davidenko differential equation (1953):

$$H = 0 \quad \longrightarrow \quad \frac{d}{dt} H = 0 \quad \longrightarrow \quad \dot{x}(t) = -J_x H(x(t), t)^{-1} J_t H(x(t), t)$$

  - Constant, Euler, Heun, Runge-Kutta, Runge-Kutta-Fehlberg, ...

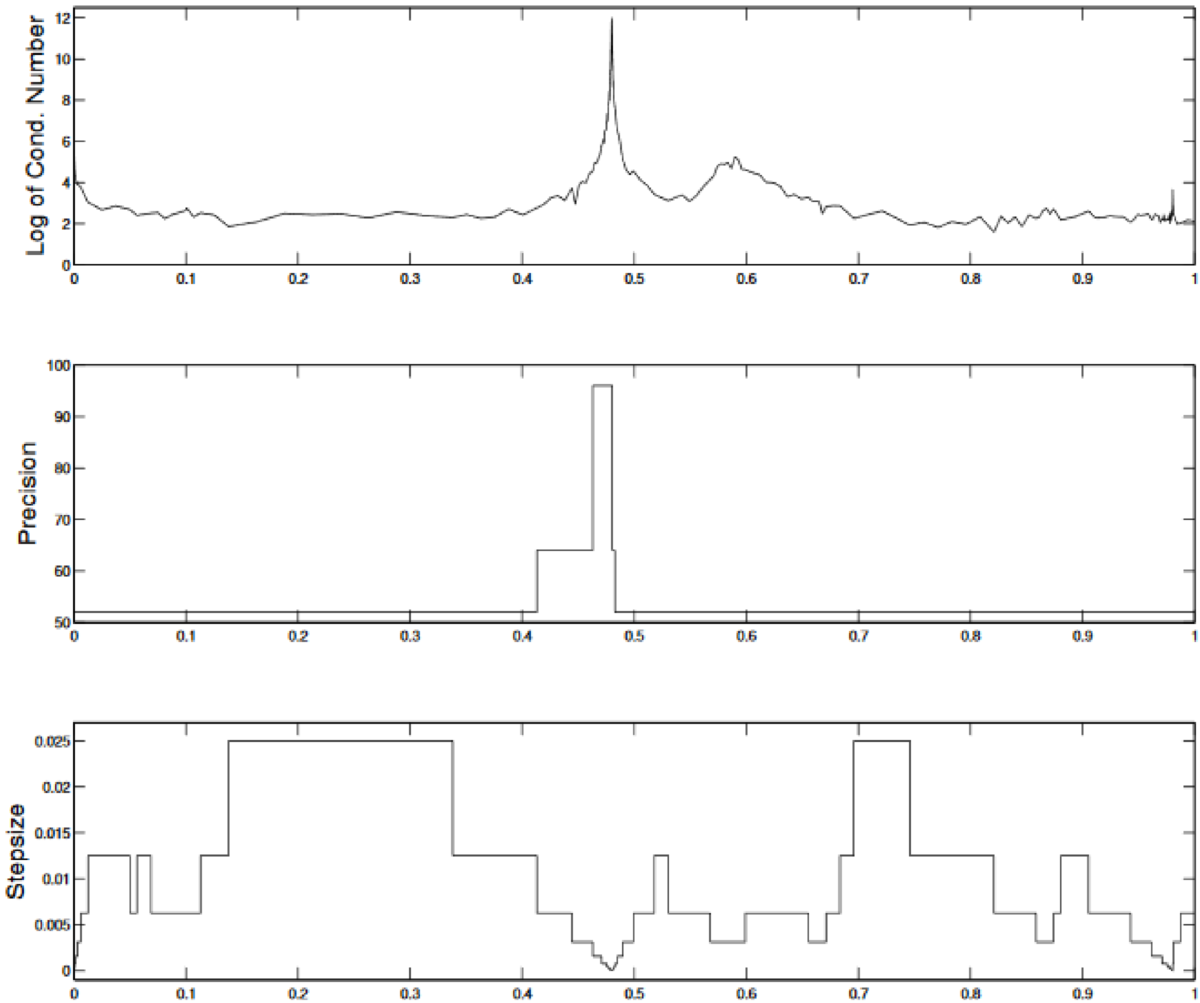
- (Corrector) for each t, apply Newton's method to $H(\bullet, t) = 0$

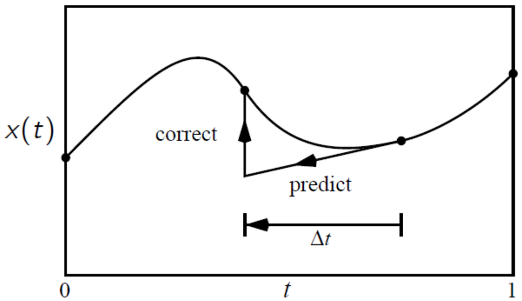


Continuation

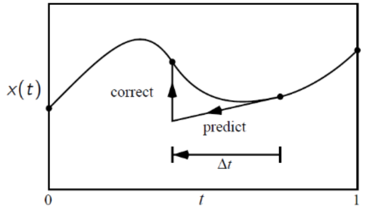
Locally adapt both stepsize and floating-point precision:

- Bates-H.-Sommese-Wampler (2008,2009), Bates-H.-Sommese (2011)





Continuation



Certified tracking (select stepsize to guarantee to track path):

- Shub-Smale ("Bézout series" 1990s), Beltran-Leykin (2011,2012), H.-Liddell (2016), Xu-Burr-Yap (2018), ...

Smale's 17^{th} problem: polynomial time to compute a root

- Beltran-Pardo (2009, 2011), Cucker-Bürgisser (2011), Lairez (2017)

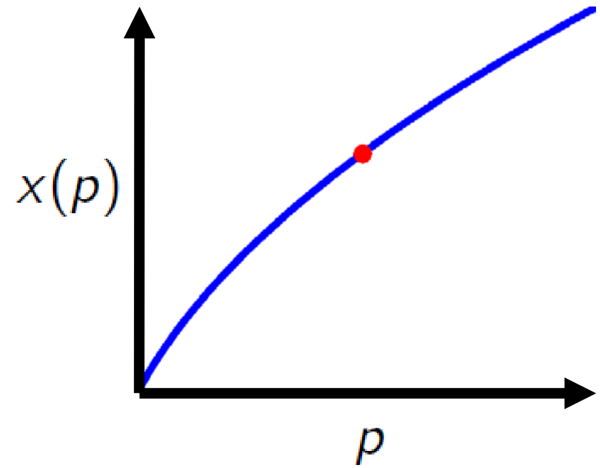
Pierre Lairez: 2017 SIAG/AG Early Career Prize

Continuation

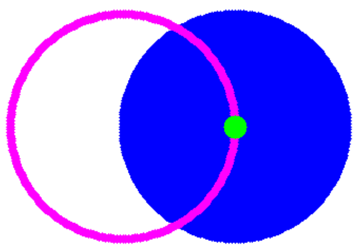
Example

$$f(x;p) = x^2 - p = 0$$

Track around a loop: $\quad x(e^{i\theta})$



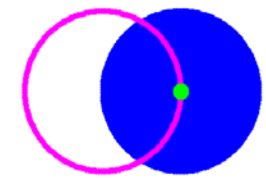
$p \in \mathbb{C}$



Example

Continuation

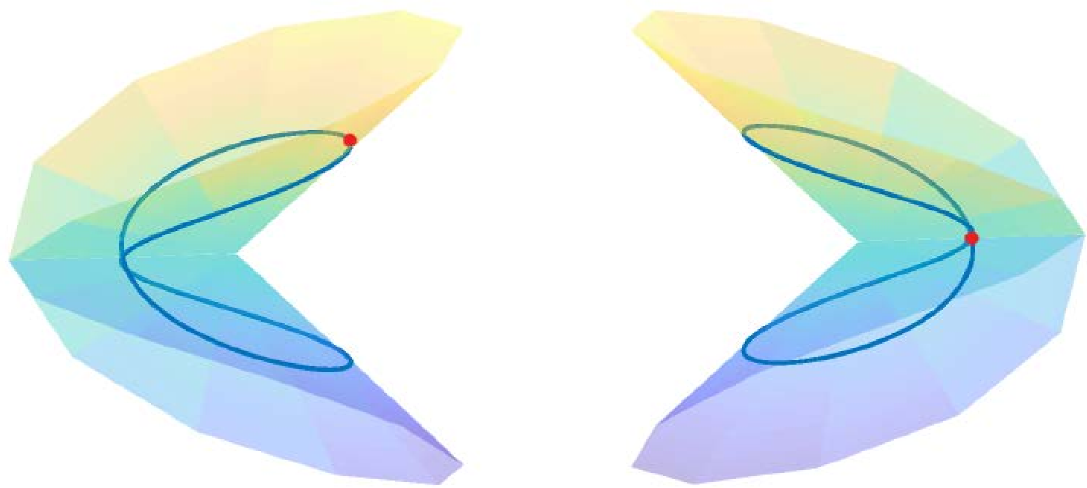
$p \in \mathbb{C}$



$$f(x;p) = x^2 - p = 0$$

Track around a loop: $x(e^{i\theta})$

- $\theta = 0$: $x = 1$
- $\theta = 2\pi$: $x = -1$
- $\theta = 4\pi$: $x = 1$



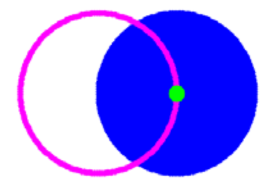
real(x) imag(x)

cycle number = winding number = 2

Continuation

Example

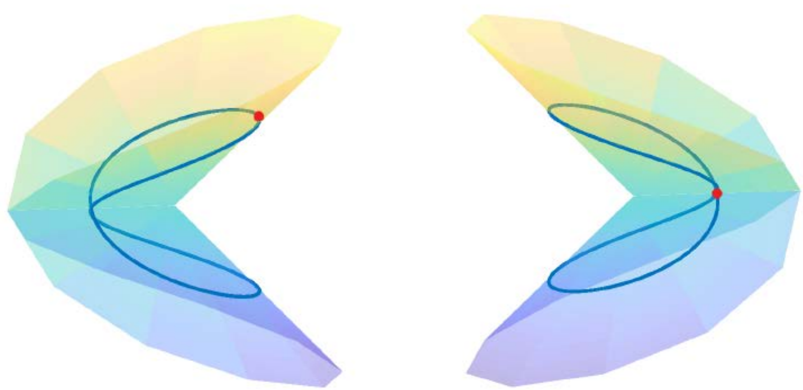
$p \in \mathbb{C}$



$$f(x;p) = x^2 - p = 0$$

Track around a loop: $\quad x(e^{i\theta})$

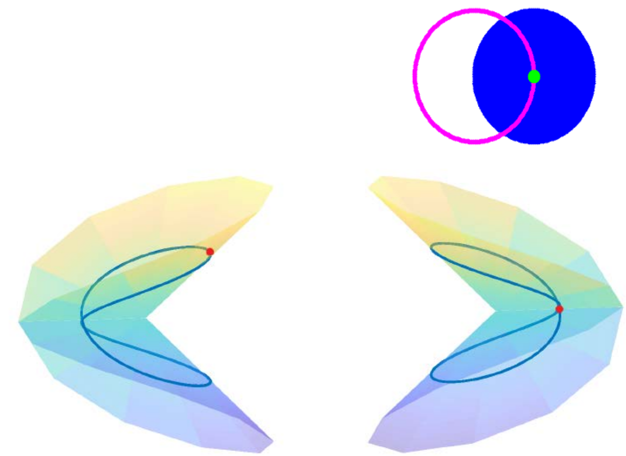
- monodromy action: permutation of solutions along loop
  - compute other solutions
    - Duff-Hill-Jensen-Lee-Leykin-Sommars (2018), Bliss-Duff-Leykin-Sommars (2018)
  - decompose solution sets



real(x) imag(x)

Continuation

Example



$$f(x;p) = x^2 - p = 0$$

Track around a loop: $x(e^{i\theta})$

- Cauchy integral theorem: computing singular endpoints
  - cycle number c
  - sufficiently small radius $r > 0$

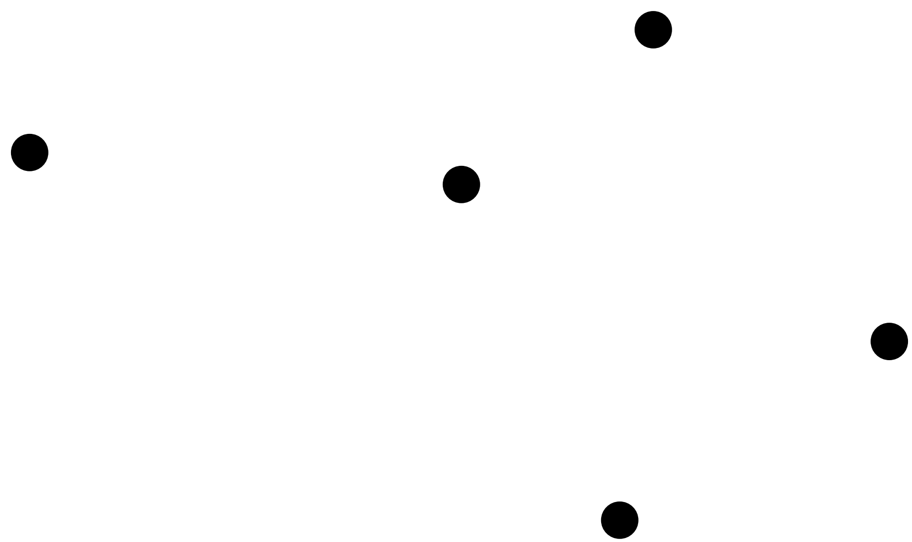
$$x(0) = \frac{1}{2\pi c}\int_0^{2\pi c} x(re^{i\theta})d\theta$$

  - Cauchy endgame: Morgan-Sommese-Wampler (1991)

Isolated Solutions

Find all isolated solutions of

$$f(x) = \begin{bmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_n(x_1, \ldots, x_n) \end{bmatrix} = 0$$

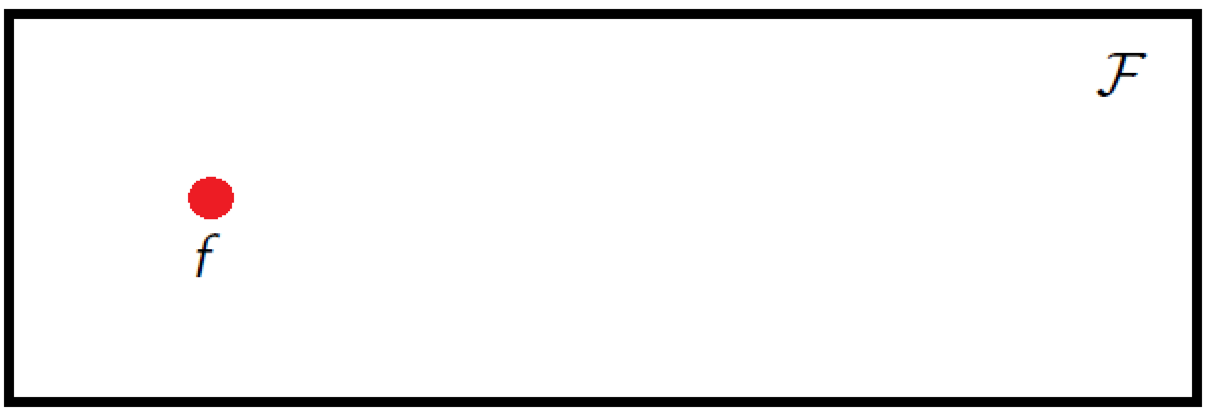


Isolated Solutions

$$f(x) = \begin{bmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_n(x_1, \ldots, x_n) \end{bmatrix} = 0$$

Homotopy continuation requires (Morgan-Sommese (1989)):

1. parameters to "continue"
   - think of f as a member of a family $\mathcal{F}$



Isolated Solutions

Homotopy continuation requires (Morgan-Sommese (1989)):

1. parameters to "continue"
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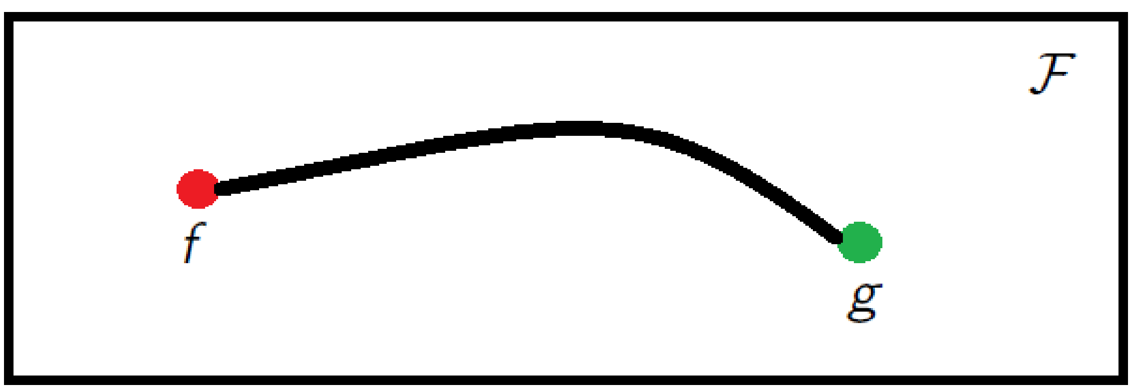
2. homotopy that describes the deformation of the parameters
   - construct a deformation inside of $\mathcal{F}$ that ends at f



Isolated Solutions

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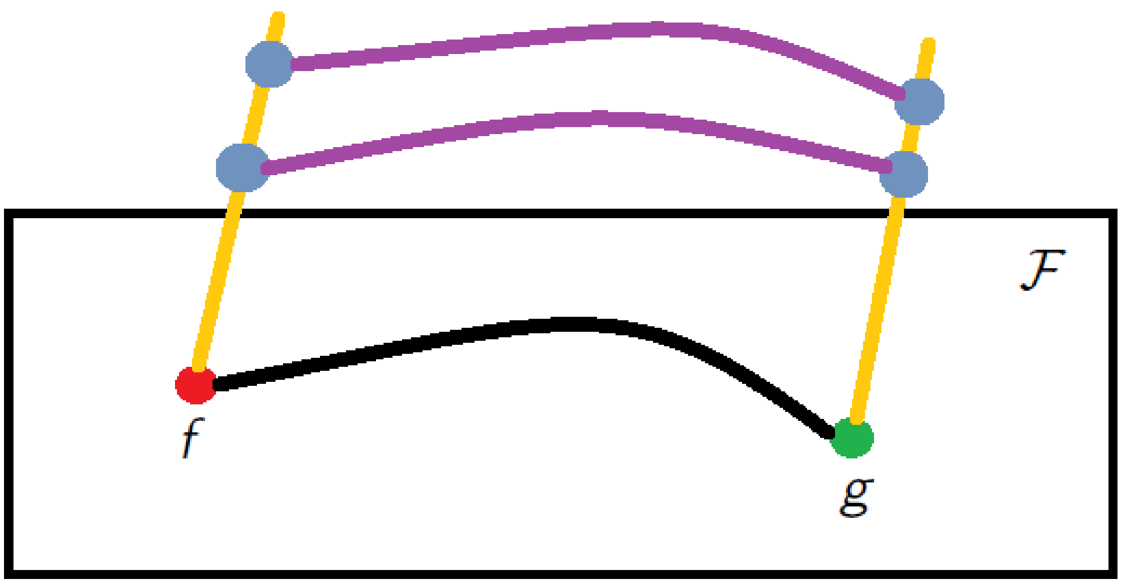
   - think of f as a member of a family $\mathcal{F}$
2. homotopy that describes the deformation of the parameters
   - construct a deformation inside of $\mathcal{F}$ that ends at f
3. start points to track along paths as parameters deform
   - parallelize computation – track each path independently

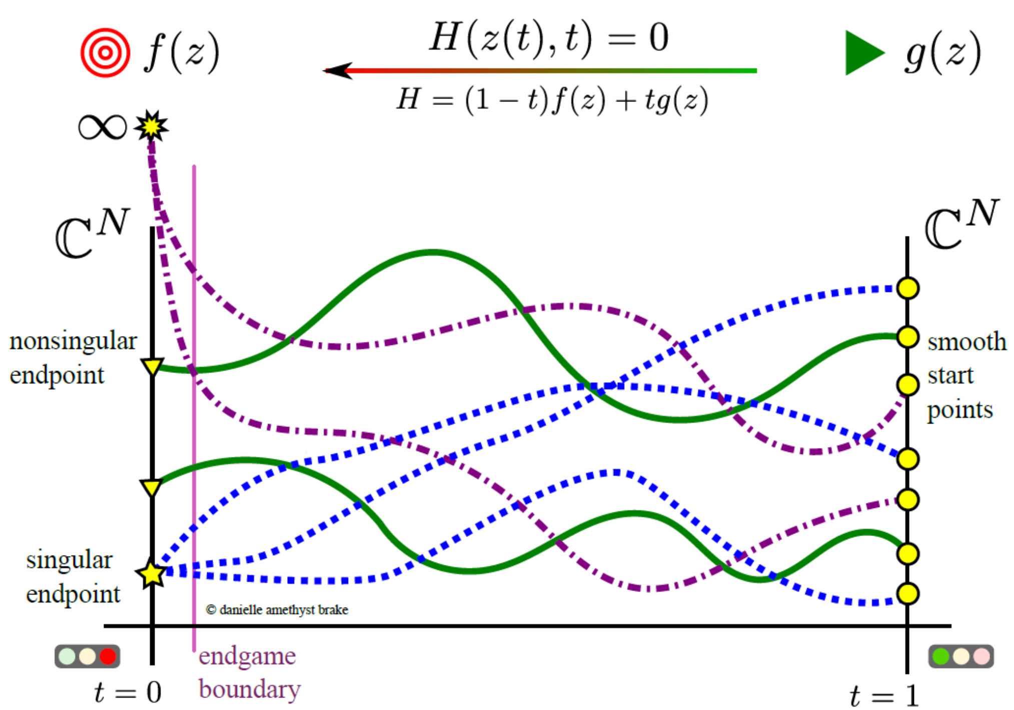


Isolated Solutions

Theorem

For properly constructed homotopies, with finite endpoints $S \subset \mathbb{C}^n$:

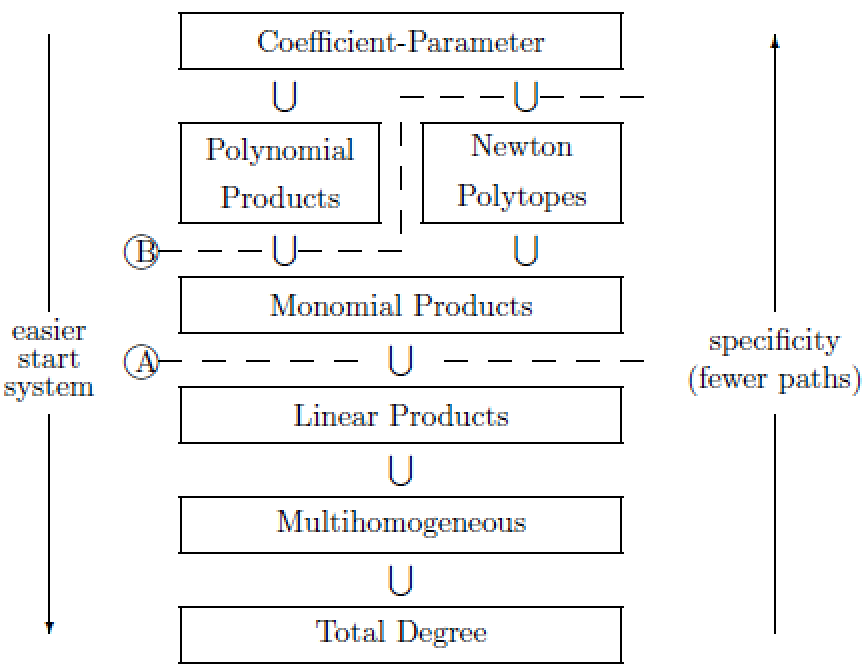
- *each isolated solution is contained in S*
  - *in fact, S contains a point on every connected component*
- *for square systems, multiplicity = number of paths if isolated.*

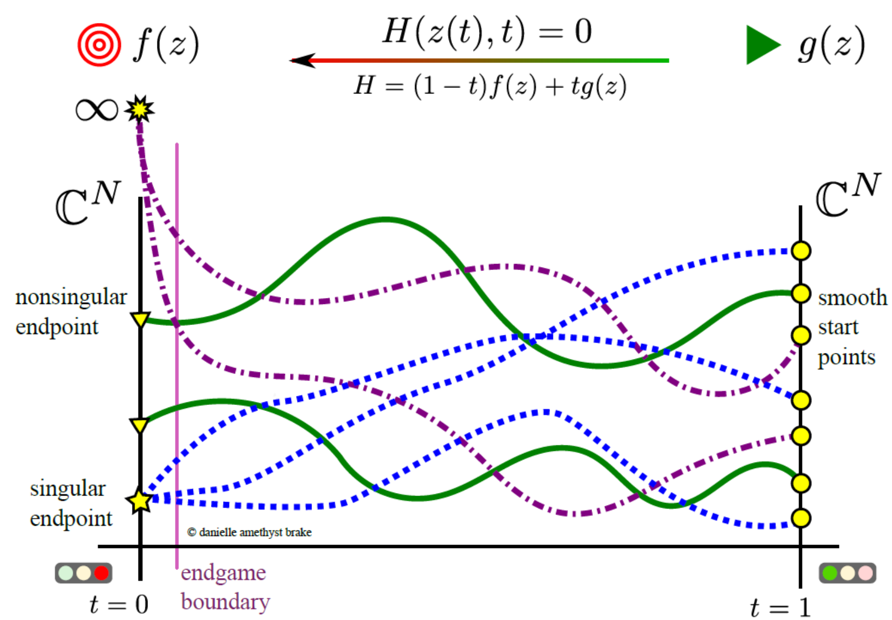


Isolated Solutions

Art in the construction of family $\mathcal{F}$:

- number of start points
- ease to compute start points





Each method is sharp for generic members of $\mathcal{F}$.

Isolated Solutions

Example

$$f = \begin{bmatrix} x^2 + 2x - 8 \\ xy + 2x + 4y - 3 \end{bmatrix}$$

Isolated Solutions

Example

$$f = \begin{bmatrix} x^2 + 2x - 8 \\ xy + 2x + 4y - 3 \end{bmatrix}$$

- Bézout family (total degree):

$$\mathcal{F} = \left\{ \begin{bmatrix} g_1(x,y) \\ g_2(x,y) \end{bmatrix} : \deg g_i = 2 \right\} \qquad g = \begin{bmatrix} x^2 - 1 \\ y^2 - 1 \end{bmatrix}$$

Number of paths = number of isolated solutions for g: 4

$$H = (1-t)\cdot f + \gamma t \cdot g$$

- $\gamma \in \mathbb{C}$ is used to create a general deformation
  - avoid singularities that arise from tracking over real numbers

Isolated Solutions

Example

$$f = \begin{bmatrix} x^2 + 2x - 8 \\ xy + 2x + 4y - 3 \end{bmatrix}$$

- Bézout family (total degree):

$$\mathcal{F} = \left\{ \begin{bmatrix} g_1(x,y) \\ g_2(x,y) \end{bmatrix} : \deg g_i = 2 \right\} \qquad g = \begin{bmatrix} x^2 - 1 \\ y^2 - 1 \end{bmatrix}$$

Number of paths = number of isolated solutions for g: 4

Bertini

```
input
  variable_group x,y;
  function f1,f2;
  f1 = x^2 + 2*x - 8;
  f2 = x*y + 2*x + 4*y - 3;
```

finite_solutions

```
1

2.000000000000000e+00 0.000000000000000e+00
-1.666666666666667e-01 0.000000000000000e+00
```

Isolated Solutions

Example

$$f = \begin{bmatrix} x^2 + 2x - 8 \\ xy + 2x + 4y - 3 \end{bmatrix}$$

- Multihomogeneous Bézout family (Morgan-Sommese (1987)):

$$\mathcal{F} = \left\{ \begin{bmatrix} g_1(x) \\ g_2(x, y) \end{bmatrix} : \begin{array}{l} \deg_x g_1 = 2, \\ \deg_x g_2 = \deg_y g_2 = 1 \end{array} \right\}$$

$$g = \begin{bmatrix} x^2 - 1 \\ (x-2)(y-1) \end{bmatrix} \qquad H = (1-t) \cdot f + \gamma t \cdot g$$

Number of paths = number of isolated solutions for g: 2

Isolated Solutions

Example

$$f = \begin{bmatrix} x^2 + 2x - 8 \\ xy + 2x + 4y - 3 \end{bmatrix}$$

- Multihomogeneous Bézout family (Morgan-Sommese (1987)):

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Number of paths = number of isolated solutions for g: 2

Bertini

input
```
variable_group x;
variable_group y;
function f1,f2;
f1 = x^2 + 2*x - 8;
f2 = x*y + 2*x + 4*y - 3;
```

Isolated Solutions

Example

$$f = \begin{bmatrix} x^2 + 2x - 8 \\ xy + 2x + 4y - 3 \end{bmatrix}$$

- Polyhedral (BKK, Huber-Sturmfels (1995)):

$$\mathcal{F} = \left\{ \begin{bmatrix} a_1 x^2 + a_2 x + a_3 \\ a_4 xy + a_5 x + a_6 y + a_7 \end{bmatrix} : a_i \in \mathbb{C} \right\}$$

$$g = \begin{bmatrix} x^2 - 1 \\ y - 1 \end{bmatrix} \qquad H = (1-t) \cdot f + \gamma t \cdot g$$

Number of paths = number of isolated solutions for g: 2

Isolated Solutions

Example

$$f = \begin{bmatrix} x^2 + 2x - 8 \\ xy + 2x + 4y - 3 \end{bmatrix}$$

- Extra structure in the coefficients of f.

$$\mathcal{F} = \left\{ p(x, y; a) = \begin{bmatrix} x^2 - (a_1 + a_2)x + a_1 a_2 \\ (x - a_1)y + a_3 x + a_4 \end{bmatrix} : a_i \in \mathbb{C} \right\}$$

$$g = \begin{bmatrix} x^2 - 1 \\ (x - 1)y - 1 \end{bmatrix}$$

Number of paths = number of isolated solutions for g: 1

Isolated Solutions

Example

$$f = \begin{bmatrix} x^2 + 2x - 8 \\ xy + 2x + 4y - 3 \end{bmatrix}$$

$$\mathcal{F} = \left\{ p(x, y; a) = \begin{bmatrix} x^2 - (a_1 + a_2)x + a_1 a_2 \\ (x - a_1)y + a_3 x + a_4 \end{bmatrix} : a_i \in \mathbb{C} \right\}$$

$$g = \begin{bmatrix} x^2 - 1 \\ (x - 1)y - 1 \end{bmatrix}$$

Since $\mathcal{F}$ is no longer linear, use a parameter homotopy:

$$H = p(x, y; a(t))$$

where $\quad a(t) = (1 - \tau(t))(-4, 2, 2, -3) + \tau(t)(1, -1, 0, -1)$

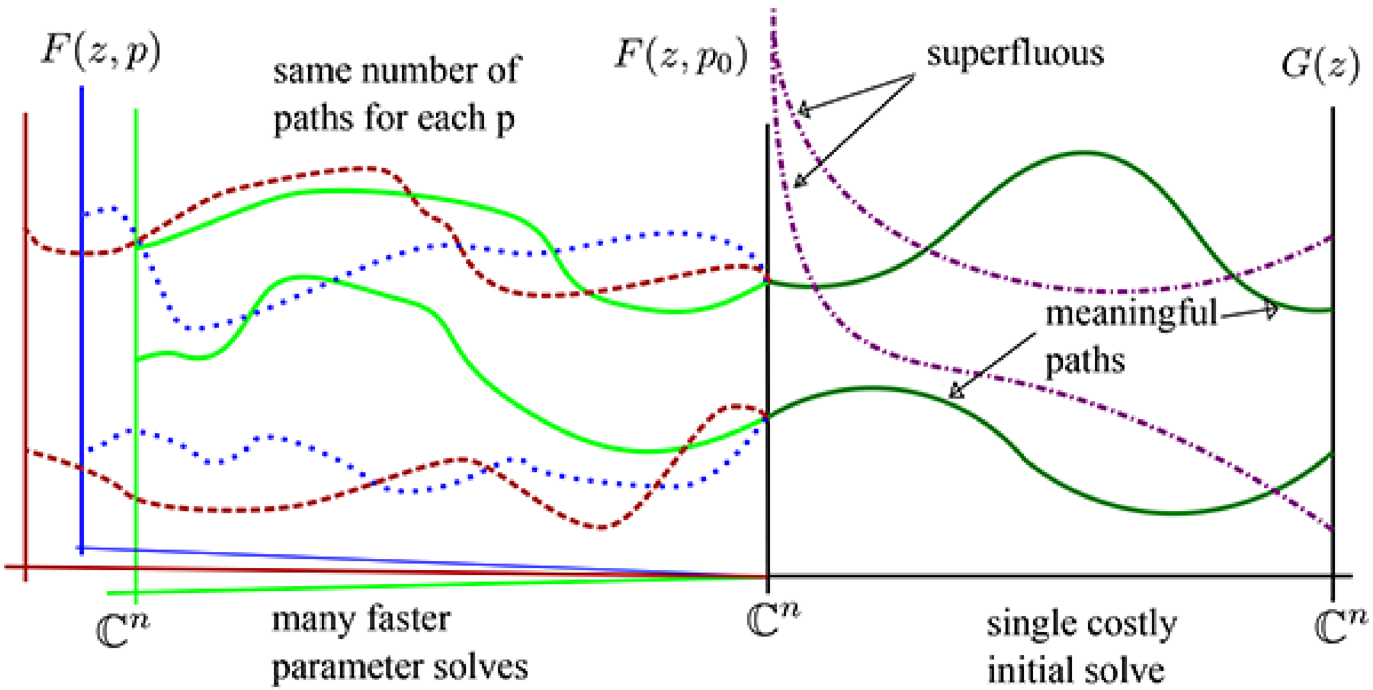
$$\tau(t) = \frac{\gamma t}{1 - t + \gamma t}$$

Isolated Solutions

Example

$$f = \begin{bmatrix} x^2 + 2x - 8 \\ xy + 2x + 4y - 3 \end{bmatrix}$$

$$\mathcal{F} = \left\{ p(x, y; a) = \begin{bmatrix} x^2 - (a_1 + a_2)x + a_1 a_2 \\ (x - a_1)y + a_3 x + a_4 \end{bmatrix} : a_i \in \mathbb{C} \right\}$$



Isolated Solutions

Some software options:

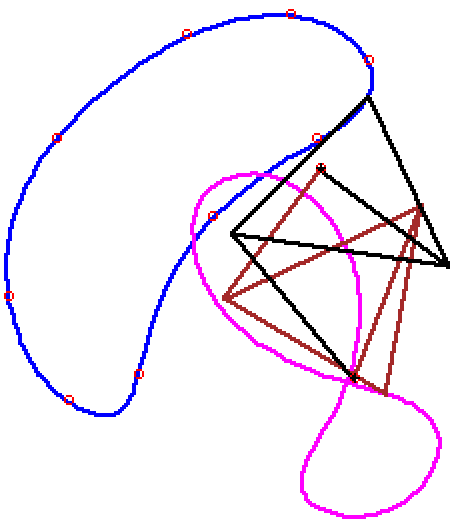
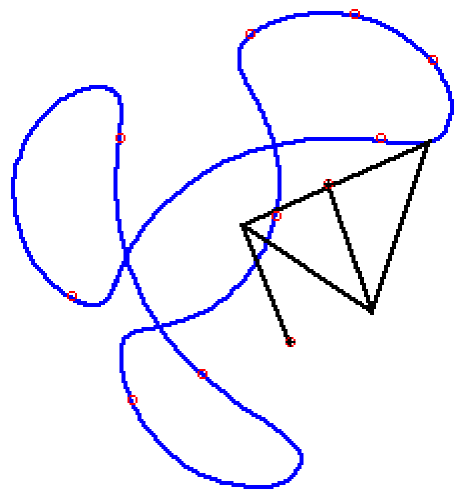
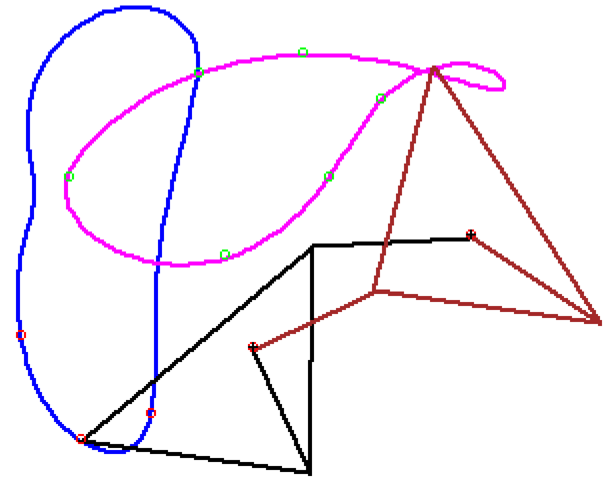
- `Bertini`
- `Bertini.m2`
- `Hom4PS`
- `HomotopyContinuation.jl`
- `MonodromySolver`
- `NAG4M2`
- `Paramotopy`
- `PHCpack`

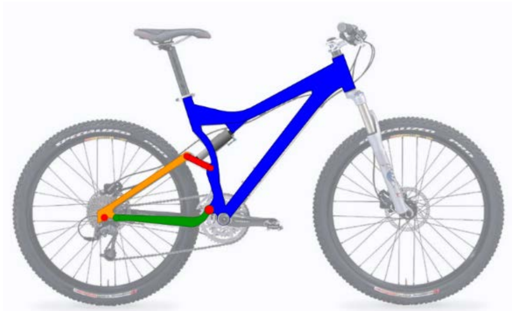
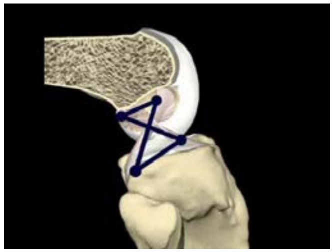
Visitors to ICERM: Bates, Brake, Chen, Duff, Hill, Lee, Leykin, Rodriguez, Sommars, Sommese, Wampler, ...

Isolated Solutions

Example (Alt's problem (1923))

Find all 4-bar linkages whose coupler curve passes through 9 given general points in the plane.

Isolated Solutions

Example (Alt's problem (1923))

Find all 4-bar linkages whose coupler curve passes through 9 given general points in the plane.

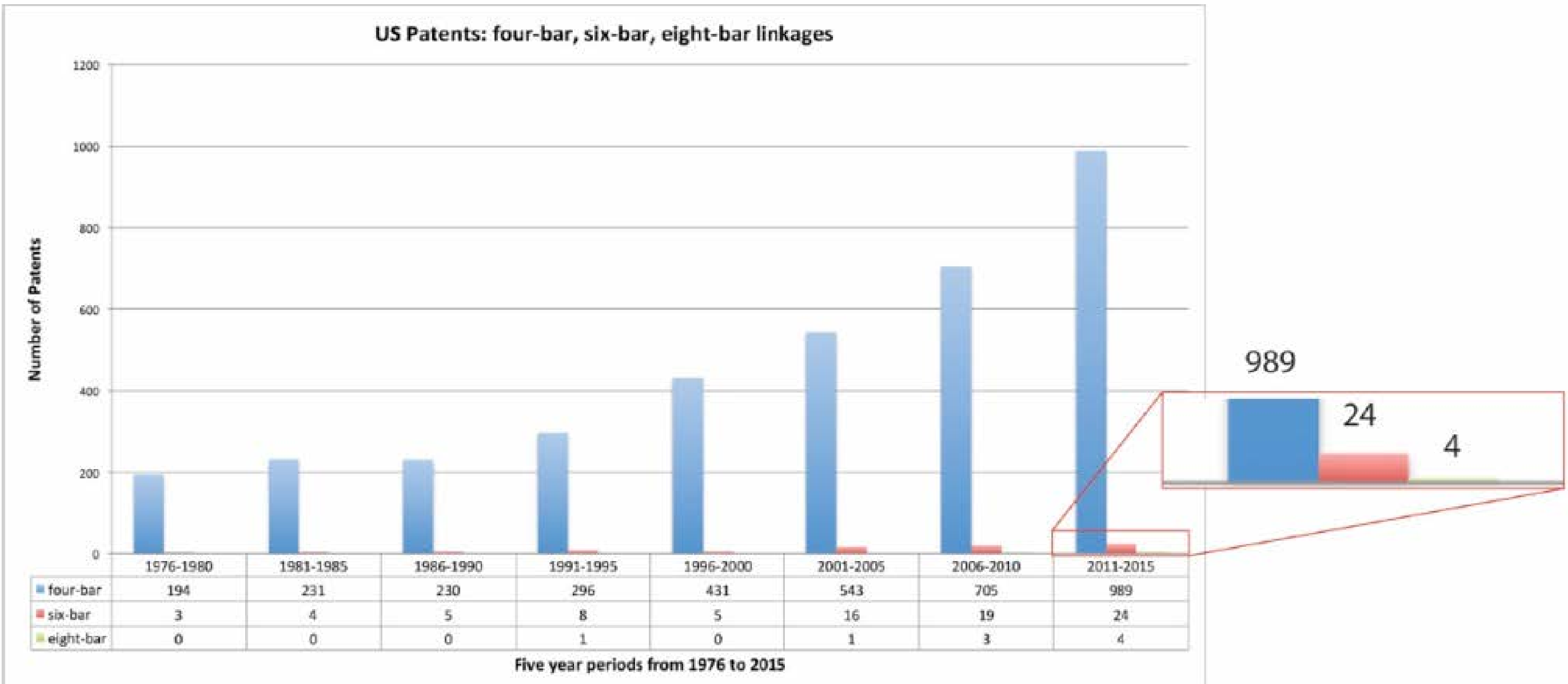
- $8652 = 6 \cdot 1442$ (Wampler-Morgan-Sommese (1992))

Their polynomial system: 4 quadratics and 8 quartics

Bézout	1,048,576	$= 2^4 \cdot 4^8$
M-hom Bézout	286,720	$= 2^{12} \cdot \binom{8}{4}$
Polyhedral	79,135	
Product decomp.	18,700	
Actual	8,652	

Isolated Solutions

Mechanical Design 101 | MECHANICAL DESIGN EDUCATIONAL RESOURCE

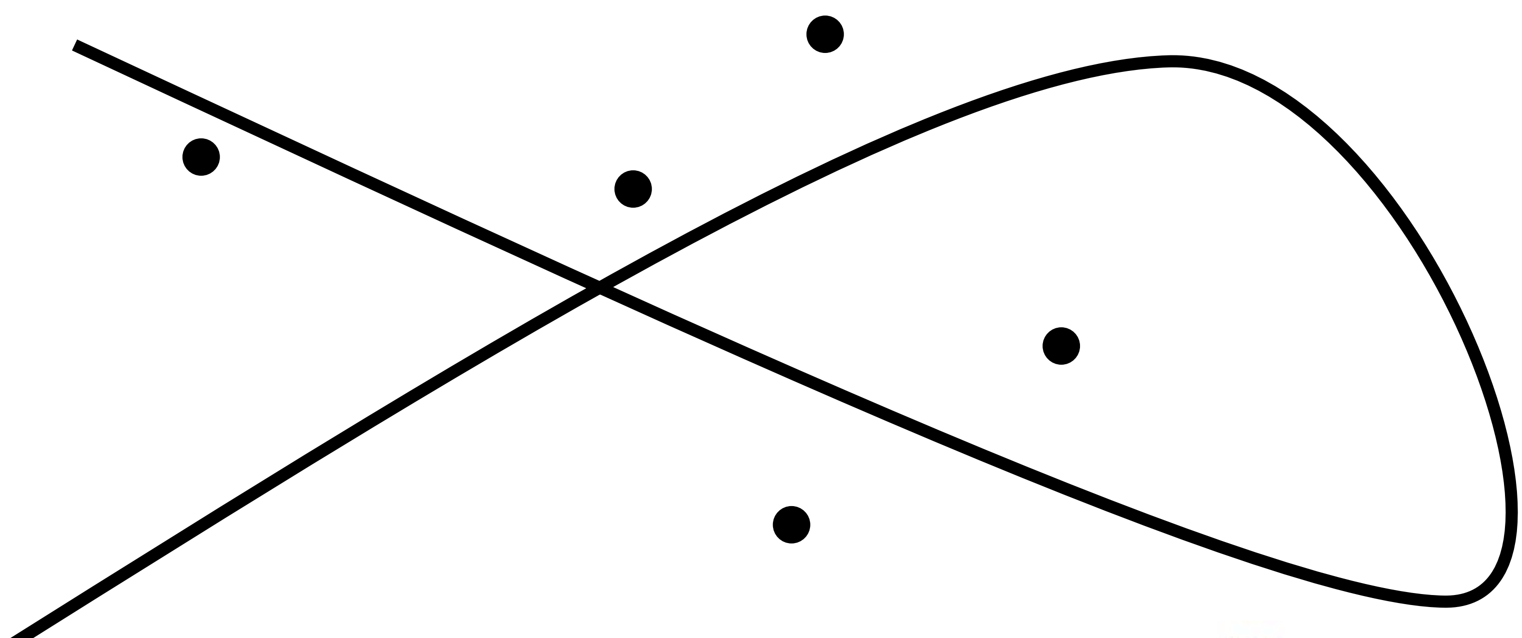


	1976-1980	1981-1985	1986-1990	1991-1995	1996-2000	2001-2005	2006-2010	2011-2015
four-bar	194	231	230	296	431	543	705	989
six-bar	3	4	5	8	5	16	19	24
eight-bar	0	0	0	1	0	1	3	4

Witness Set

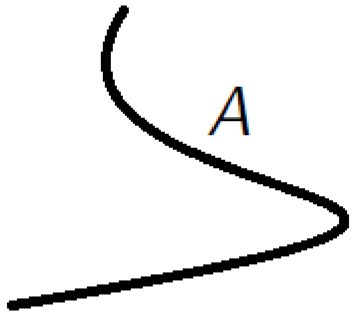
Describe all solutions of

$$f(x) = \begin{bmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_k(x_1, \ldots, x_n) \end{bmatrix} = 0$$



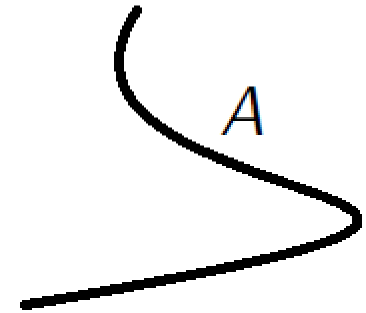
Witness Set

How to represent an irreducible algebraic variety A on a computer?



Witness Set

How to represent an irreducible algebraic variety A on a computer?



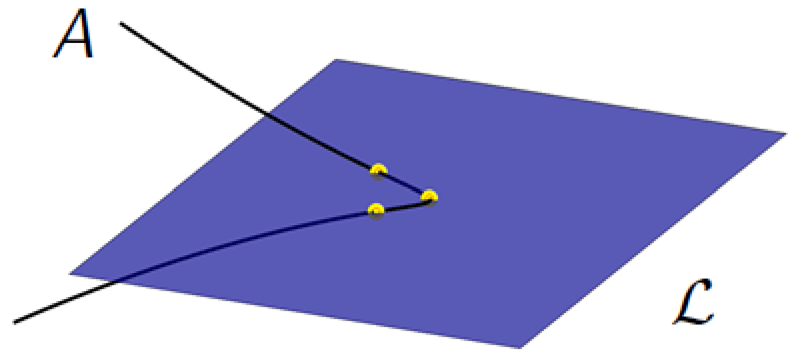
- algebraic: prime ideal $I(A) = \{g \mid g(a) = 0 \text{ for all } a \in A\}$
  - Hilbert Basis Theorem (1890): there exists $f_1, \ldots, f_k$ such that

$$I(A) = \langle f_1, \ldots, f_k \rangle$$

Witness Set

How to represent an irreducible algebraic variety A on a computer?

- geometric: witness set $\{f, \mathcal{L}, W\}$ where
  - f is polynomial system where A is an irred. component of $\mathcal{V}(f)$
  - $\mathcal{L}$ is a linear space with $\mathrm{codim}\ \mathcal{L} = \dim\ A$
  - $W = \mathcal{L} \cap A$ where $\#W = \deg\ A$

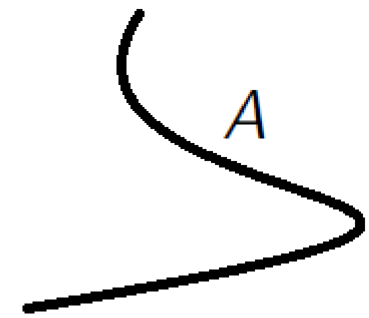


- Witness sets "localize" computations to A effectively ignoring other irreducible components
- Sample points from A by moving the linear slice $\mathcal{L}$

Witness Set

Example

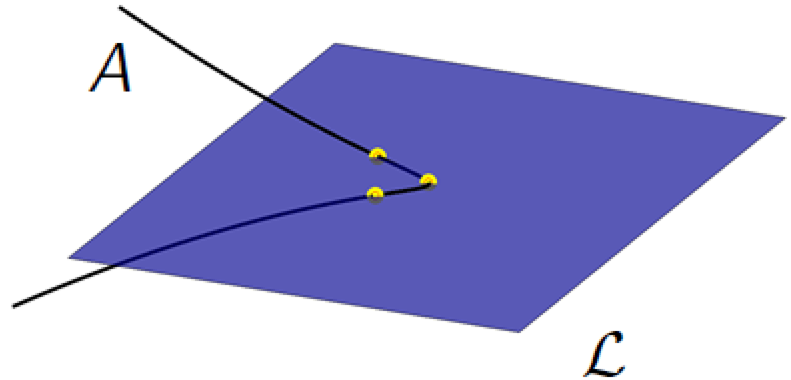
$$A = \{[s^3, s^2t, st^2, t^3] \mid [s,t] \in \mathbb{P}^1\} \subset \mathbb{P}^3$$ – twisted cubic curve



Witness Set

Example

$$A = \{[s^3, s^2t, st^2, t^3] \mid [s,t] \in \mathbb{P}^1\} \subset \mathbb{P}^3 \text{ – twisted cubic curve}$$

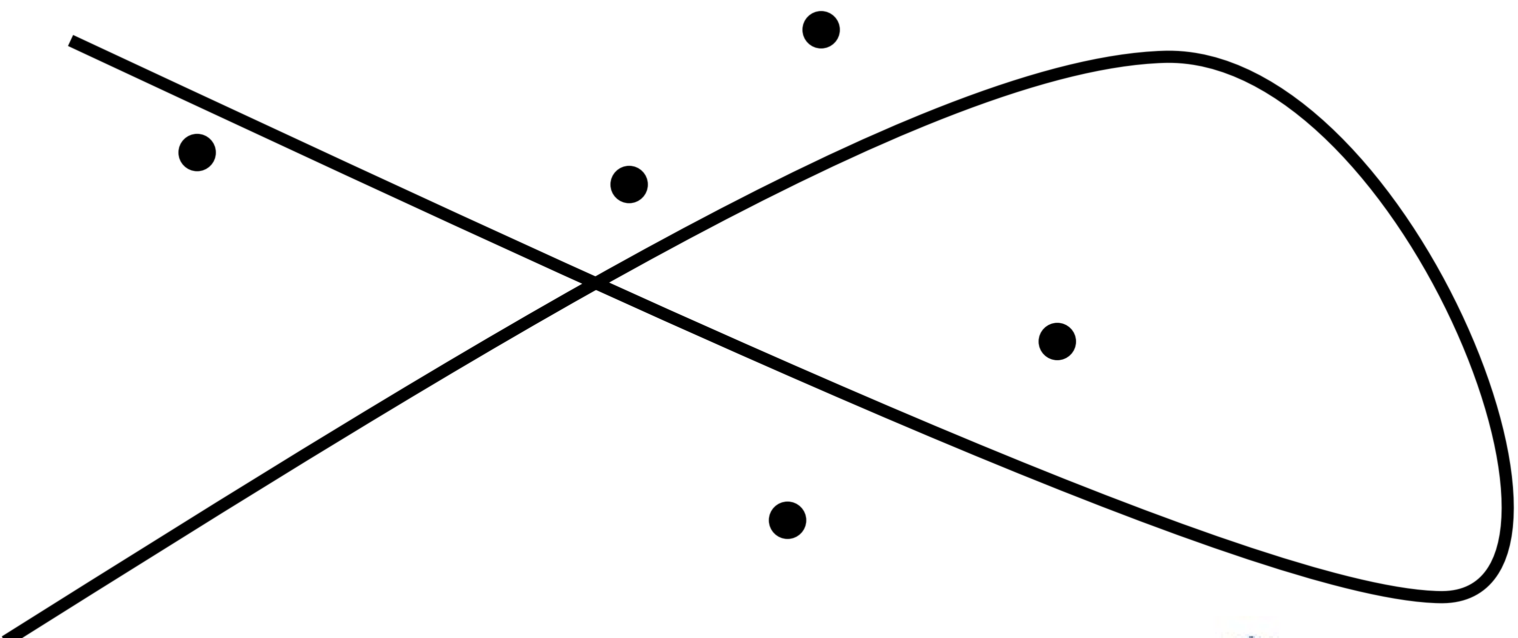


- $\{f, \mathcal{L}, W\}$ where
  - $f = \begin{bmatrix} x_1^2 - x_0x_2 \\ x_1x_2 - x_0x_3 \end{bmatrix}$
  - $\mathcal{L} = \{[x_0, x_1, x_2, x_3] \in \mathbb{P}^3 \mid 6x_0 - 6x_1 - 2x_2 + x_3 = 0\} \subset \mathbb{P}^3$
    - $\operatorname{codim} \mathcal{L} = \dim A = 1$
  - $W = \left\{ \begin{array}{c} [1, 3.2731, 10.7130, 35.0644], \\ [1, 0.8596, 0.7389, 0.6351], \\ [1, -2.1326, 4.5481, -9.6995] \end{array} \right\}$
    - $\deg A = 3$

Witness Set

Numerical irreducible decomposition:

- compute a witness set for each irreducible component



Witness Set

Example

- $f = \begin{bmatrix} x_1^2 - x_0x_2 \\ x_1x_2 - x_0x_3 \end{bmatrix}$

Bertini

input

```
CONFIG
 TrackType: 1;
END;
INPUT
 hom_variable_group x0,x1,x2,x3;
 function f1,f2;
 f1 = x1^2 - x0*x2;
 f2 = x1*x2 - x0*x3;
END;
```

```
Dimension 1: 2 classified components
-----------------------------------------
   degree 1: 1 component
   degree 3: 1 component
```

Witness Set

Reduce to codimension $=$ # equations via randomization:

Theorem (Bertini)

Let $f : \mathbb{C}^n \to \mathbb{C}^N$ *and* $A \subset \mathcal{V}(f) \subset \mathbb{C}^n$ *be an irreducible component with* $\operatorname{codim} A = c$*. If* $R \in \mathbb{C}^{c \times N}$ *is general, then*

- A *is an irreducible component of* $V(R \cdot f)$
- $V(R \cdot f) \setminus V(f)$ *is either empty or smooth of codimension* c*.*

Witness Set

Reduce to codimension = # equations via randomization:

Theorem (Bertini)

Let $f : \mathbb{C}^n \to \mathbb{C}^N$ and $A \subset \mathcal{V}(f) \subset \mathbb{C}^n$ be an irreducible component with $\operatorname{codim} A = c$. If $R \in \mathbb{C}^{c \times N}$ is general, then

- *A is an irreducible component of $V(R \cdot f)$*
- *$V(R \cdot f) \setminus V(f)$ is either empty or smooth of codimension c.*

Example

For general $R \in \mathbb{C}^{2\times 3}$ and $f = \begin{bmatrix} x_1^2 - x_0x_2 \\ x_1x_2 - x_0x_3 \\ x_2^2 - x_1x_3 \end{bmatrix}$,

- $V(R \cdot f) =$ twisted cubic + line

Witness Set

Example

$$f = \begin{bmatrix} (x-y)(\hat{x}-\hat{y}) \\ (x-y)(\hat{a}\hat{x} - 2\hat{a}\hat{y} + 2\hat{b}\hat{x} - \hat{b}\hat{y}) \\ (\hat{x}-\hat{y})(ax - 2ay + 2bx - by) \\ \hat{a}\hat{b}(x-y)(\hat{a}\hat{y} - \hat{b}\hat{x}) \\ ab(\hat{x}-\hat{y})(ay - bx) \\ \vdots \\ (a\hat{b}\hat{x}y - \hat{a}bx\hat{y})(a\hat{b}\hat{x} - \hat{a}\hat{b}x - \hat{a}b\hat{y} + \hat{a}\hat{b}y - a\hat{x}\hat{y} + \hat{a}x\hat{y} + b\hat{x}\hat{y} - \hat{b}\hat{x}y) \\ (a\hat{b}\hat{x}y - \hat{a}bx\hat{y})(ab\hat{x} - \hat{a}bx - ab\hat{y} + a\hat{b}y - a\hat{x}y + \hat{a}xy + bx\hat{y} - \hat{b}xy) \end{bmatrix}$$

15 polynomials in 8 variables $a, b, x, y, \hat{a}, \hat{b}, \hat{x}, \hat{y}$

For general $R \in \mathbb{C}^{8\times 15}$:

- $V(R\cdot f) \setminus V(f)$ consists of finitely many points
  - all nonsingular with respect to $R \cdot f = 0$

Witness Set

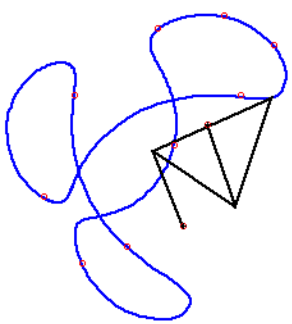
Example

$$f = \begin{bmatrix} (x-y)(\hat{x}-\hat{y}) \\ (x-y)(\hat{a}\hat{x}-2\hat{a}\hat{y}+2\hat{b}\hat{x}-\hat{b}\hat{y}) \\ (\hat{x}-\hat{y})(ax-2ay+2bx-by) \\ \hat{a}\hat{b}(x-y)(\hat{a}\hat{y}-\hat{b}\hat{x}) \\ ab(\hat{x}-\hat{y})(ay-bx) \\ \vdots \\ (a\hat{b}\hat{x}y-\hat{a}bx\hat{y})(a\hat{b}\hat{x}-\hat{a}\hat{b}x-\hat{a}b\hat{y}+\hat{a}\hat{b}y-a\hat{x}\hat{y}+\hat{a}x\hat{y}+b\hat{x}\hat{y}-\hat{b}\hat{x}y) \\ (a\hat{b}\hat{x}y-\hat{a}bx\hat{y})(ab\hat{x}-\hat{a}bx-ab\hat{y}+a\hat{b}y-a\hat{x}y+\hat{a}xy+bx\hat{y}-\hat{b}xy) \end{bmatrix}$$

15 polynomials in 8 variables $a, b, x, y, \hat{a}, \hat{b}, \hat{x}, \hat{y}$

For general $R \in \mathbb{C}^{8\times 15}$:

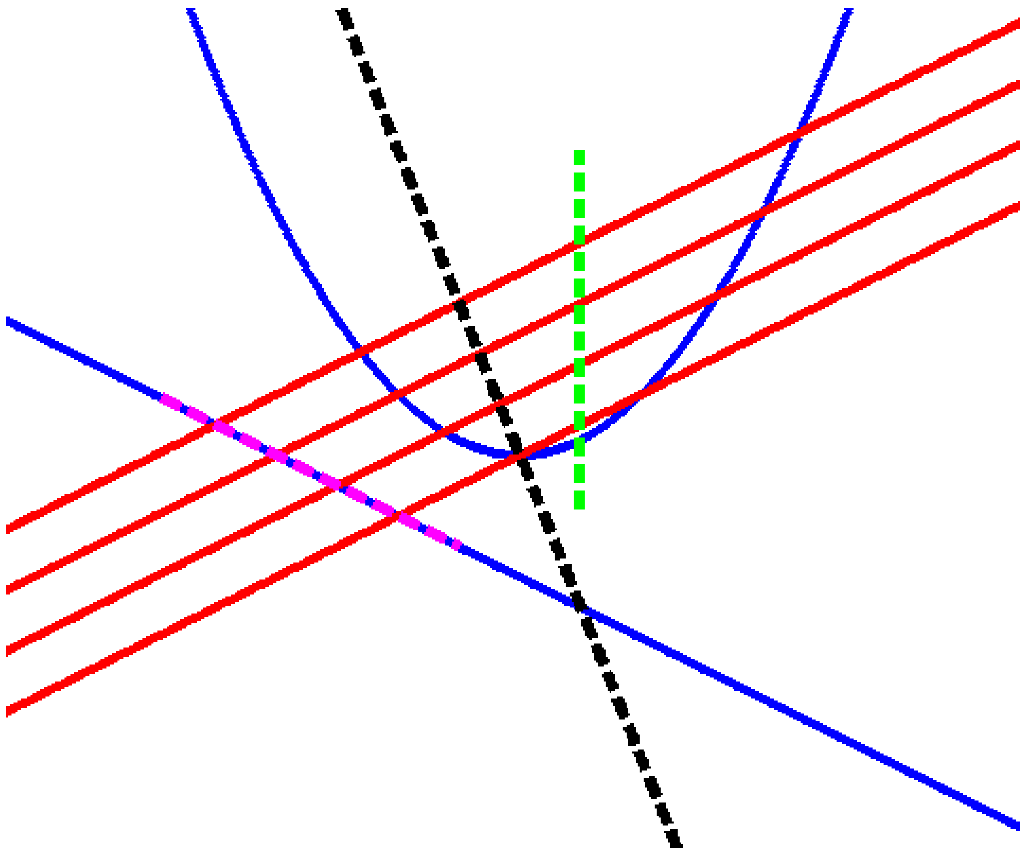
- $V(R\cdot f) \setminus V(f)$ consists of finitely many points
  - all nonsingular with respect to $R \cdot f = 0$
- Using `Bertini`: $|V(R\cdot f) \setminus V(f)| = 8652$
  - Proving this would complete proof of Alt's problem



Witness Set

Given $W \subset V(f) \cap \mathcal{L}$, how to test that $W = \mathcal{L} \cap A$ for some variety $A \subset V(f)$?

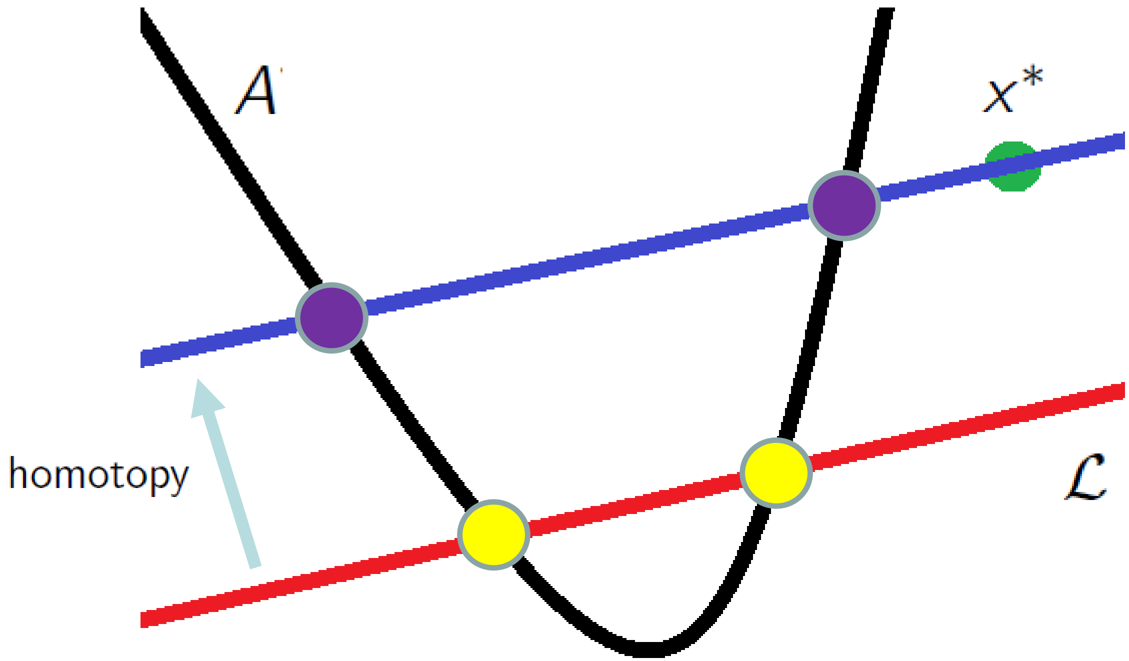
- Trace test: centroid moves linearly as slices moves in parallel



Witness Set

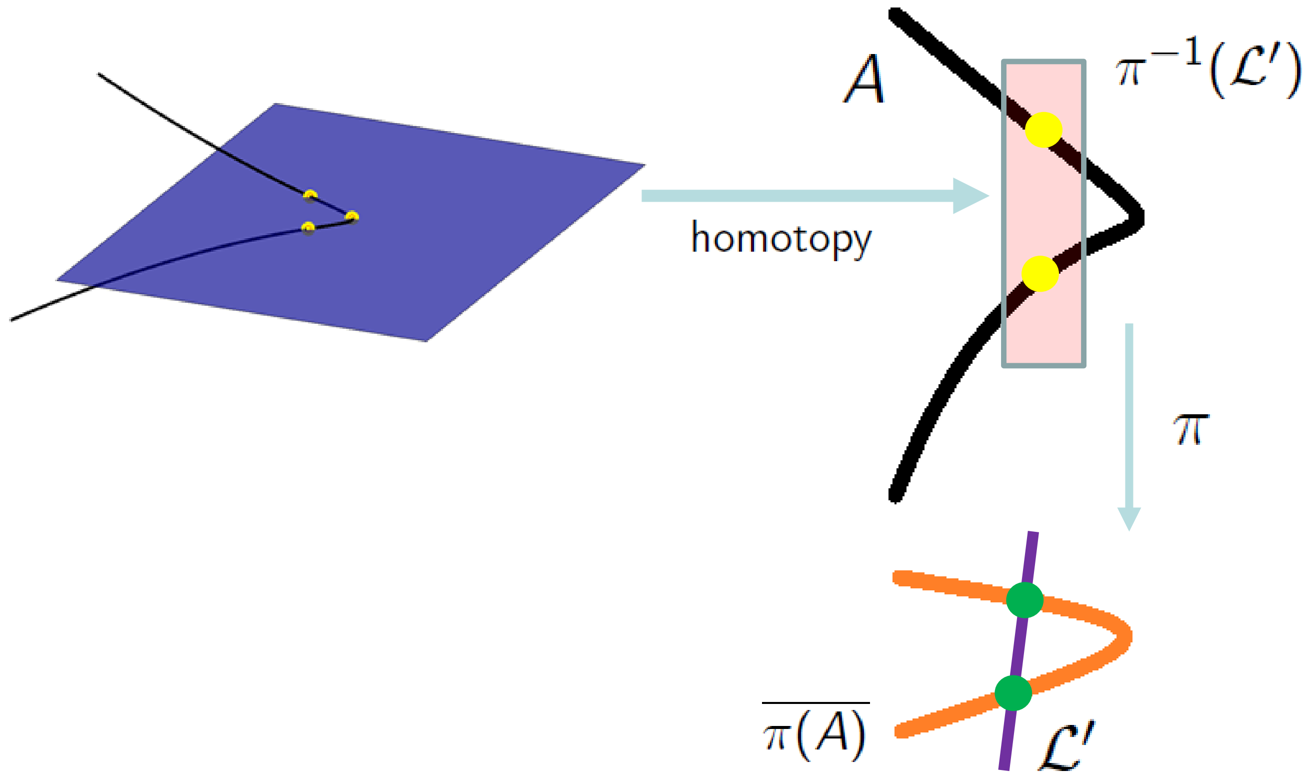
Many other numerical algebraic geometric computations can be performed starting from witness sets, such as:

- membership testing: is $x^* \in A$?
  - decide if $g(x^*) = 0$ for every $g \in I(A)$ without knowing $I(A)$



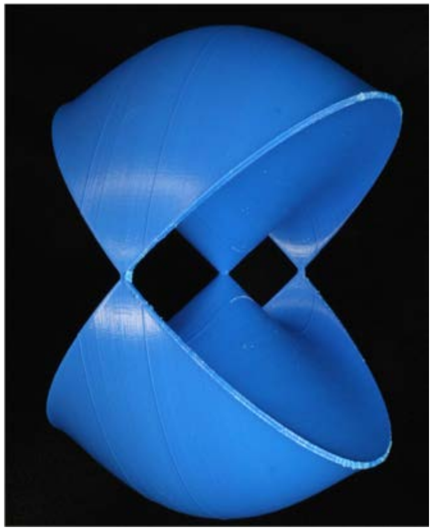
Witness Set

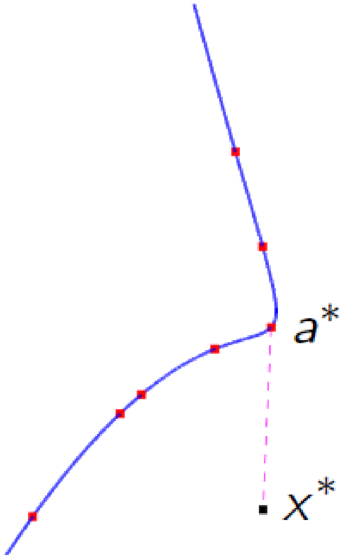
- projection: $\overline{\pi(A)}$
  - perform computations on $\overline{\pi(A)}$ without knowing *any* polynomials that vanish on $\overline{\pi(A)}$



Witness Set

- intersection: $A \cap B$
  - special case is *regeneration*
    - $\mathcal{V}(f_1, \ldots, f_k, f_{k+1}) = \mathcal{V}(f_1, \ldots, f_k) \cap \mathcal{V}(f_{k+1})$ via witness sets
  - compute A_{sing}
  - compute critical points of optimization problem



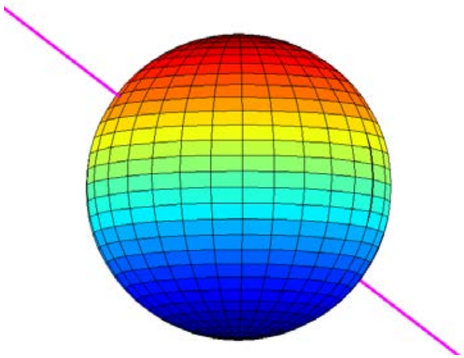


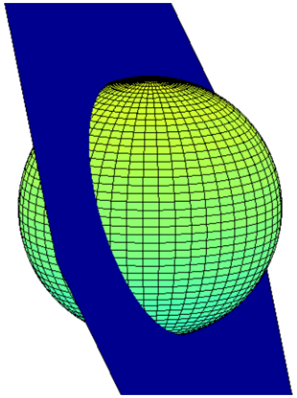
min $\|x^* - a\|_2$ such that $a \in A \cap \mathbb{R}^n$

Witness Set

Test other algebraic properties of A

- is A arithmetically Cohen Macaulay?
- is A arithmetically Gorenstein?
- is A a complete intersection?





Summary

Numerical algebraic geometry provides a toolbox for solving polynomial systems.

- "If a problem was easy, someone else would have solved it."
  - Gröbner basis computation probably did not terminate
- think carefully about what information you want/need
- art in building efficient homotopies that incorporate structure
- preconditioning is important
  - transform problem into form suitable for num. computations

Thank You!